SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

APRIL 2005

TASK #2

YEAR 12

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** section in a separate answer booklet.

Total Marks - 85 Marks

- Attempt Questions 1 - 9
- All questions are NOT of equal value.

Examiner: *E. Choy*

Total marks – 85
Attempt Questions 1 – 9
All questions are NOT of equal value

Answer each SECTION in a SEPARATE writing booklet.

Section A

Marks

Question 1 (26 marks)

(a) Differentiate

(i) $y = (5-2x)^9$; **2**

(ii) $y = x + \sqrt{x}$. **2**

(b) If $\int_2^5 f(x)\,dx = 12$, find the value of $\int_2^5 \frac{f(x)}{4}\,dx$ **1**

(c) Find the second derivative of $(x^2+3)^{-2}$ **2**

(d) Evaluate $\int_1^4 \frac{dt}{\sqrt{t}}$ **2**

(e) Find the equation of the normal to the curve $y = \frac{2}{x}$ at the point where $x = 1$. **3**

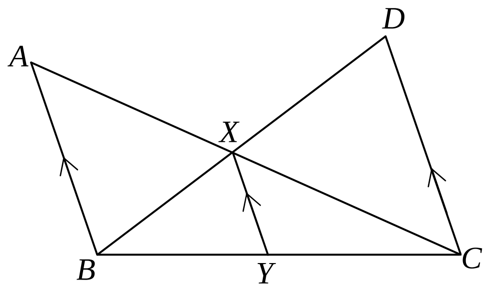
(f) Find the exact volume generated when the area under the straight line $y = x-2$ from $x = 2$ to $x = 5$ is rotated about the x axis. **2**

(g) In a certain car factory, 20% of the engines have some defect. If 20 engines are examined find the probability of at least one of them being defective. **2**

(h) In the diagram below $AB \parallel XY \parallel DC$.

$XB = 12$, $XC = 30$, $BY = 8$, $YC = 24$

$AX = a$, $DX = b$, $AB = c$, $DC = d$.



NOT TO SCALE

Copy the diagram into your answer booklet

(i) Find a and b; **2**

(ii) Find $c : d$. **2**

(i) In a raffle 20 tickets are sold and there are 2 prizes, 1st and 2nd prize.
What is the probability that a man buying 5 tickets wins

(i) the first prize; **1**

(ii) a prize; **2**

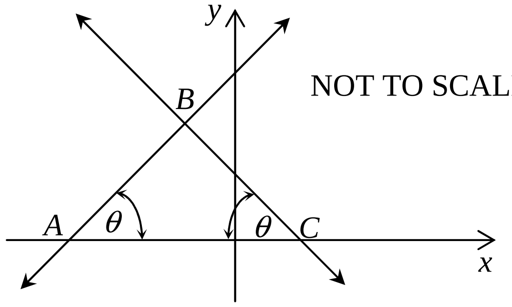
(iii) at least one prize; **2**

(iv) All 2 prizes. **1**

Section A continues over the page

Question 2 (14 marks) **Marks**

(a)



In the diagram above, the vertices of the triangle ABC are $A(-3,0)$, $B(-1,4)$ and C, where C lies on the x axis such that $\angle BAC = \angle ACB$.

Copy the diagram into your answer booklet.

(i) Find the coordinates of the midpoint of AB; **1**

(ii) Find the gradient of AB and show that $\tan\theta = 2$; **2**

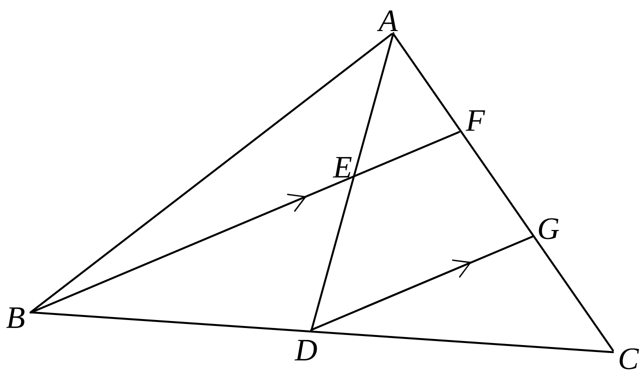
(iii) Show that the equation of AB is $y = 2x + 6$; **1**

(iv) Explain why BC has gradient -2 and hence find the equation of BC; **2**

(v) Find the coordinates of C and hence the area of $\Delta\ ABC$; **2**

(vi) Find the length of BC and hence find the perpendicular distance from A to BC. **2**

(b) In the diagram below D and E are midpoints of BC and AD respectively and $DG \parallel BF$.



Copy the diagram into your answer booklet.

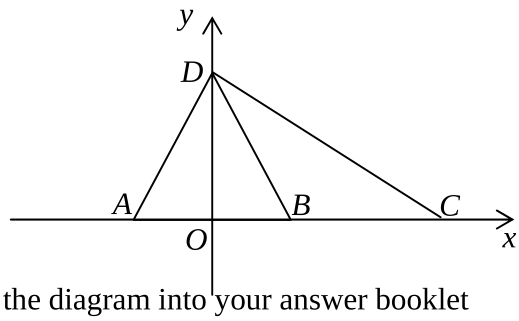
Prove $AF = FG = GC$. **4**

End of Section A

Section B (Use a SEPARATE writing booklet)

Question 3 (7 marks) **Marks**

The diagram below shows the points $A(-a,0)$, $B(a,0)$, $C(3a,0)$ and $D(0,\sqrt{3}\,a)$ on the number plane.



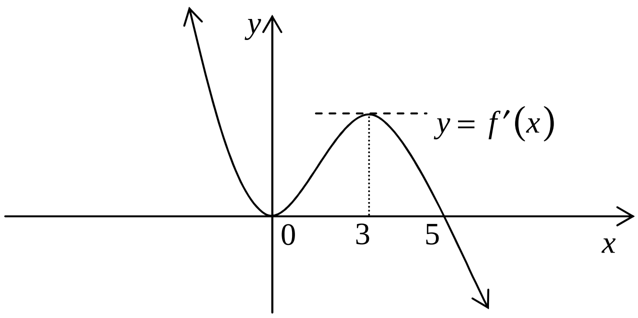
Copy the diagram into your answer booklet

(i) Show that ΔDAB is equilateral **2**

(ii) Show that ΔBCD is isosceles with $BD = BC$. **2**

(iii) Show that $BD^2 = \frac{1}{3}CD^2$. **3**

Question 4 (6 marks)



The diagram above shows the graph of the *derivative* of a certain function $f(x)$.

(i) Find the value(s) of x for which the function $f(x)$ has stationary points and determine the nature of the stationary points. **2**

(ii) Find the value(s) of x for which the function is decreasing. **2**

(iii) Find the value(s) of x for which the function $f(x)$ is concave down. **2**

Section B continues over the page

Question 5 (6 marks) **Marks**

(a) Sketch the curve $y = 2^x$. **4**

Let $J = \int_0^6 2^x \, dx$

By using the *Trapezoidal rule* with 3 sub-intervals, find an approximation to J correct to 3 significant figures.

(The *exact* value of J is $90 \cdot 9$ correct to 3 significant figures)

(b) Evaluate $\int_0^4 x\sqrt{x} \, dx$ **2**

Question 6 (5 marks)

The region under the graph $y = 3^{x+1}$ between $x = 1$ and $x = 3$ is rotated about the x axis. **5**

Using *Simpson's rule* with five function values, estimate the volume of the solid formed.

Leave your answer correct to 3 decimal places.

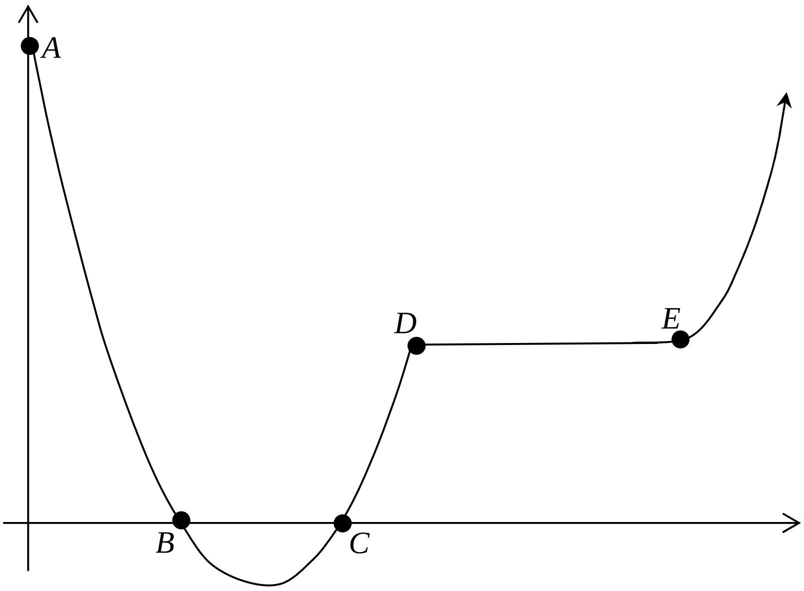
End of Section B

Section C (Use a SEPARATE writing booklet)

Question 7 (3 marks) **Marks**

The diagram below shows the graph of $y = f\left(x\right)$. **3**

At what point(s) is $f\left(x\right)$ **not** differentiable? Justify.



Question 8 (10 marks)

For the curve $y = 2x^3 - 9x^2 + 52$

(i) Find the stationary points and determine their nature. **2**

(ii) Find any points of inflexion. **2**

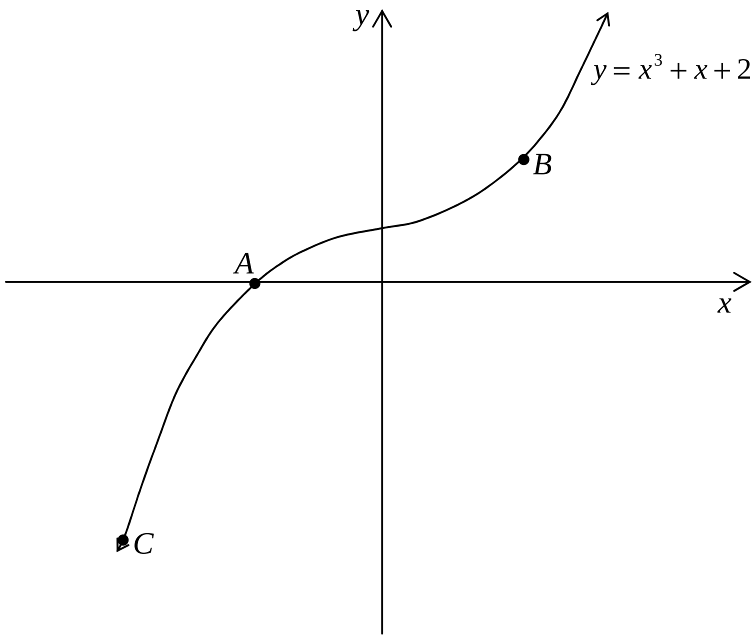
(iii) Sketch the graph of the curve. **2**

(iv) Find the maximum and minimum values of the curve in the interval $-1 \le x \le 2$. **2**

(v) Find the value(s) of k for which the equation $2x^3 - 9x^2 + 52 = k$ has *three* distinct solutions. **2**

Section C continues over the page

Question 9 (8 marks) **Marks**



In the diagram above, the curve $y = x^3 + x + 2$ cuts the x axis at $A(-1, 0)$. The tangent at another point B, parallel to the tangent at A, cuts the curve again at C.

(i) Show that the equation of the tangent at B is $y = 4x$. **3**

(ii) Show that the point C has coordinates $(-2, -8)$. **2**

(iii) Find the area enclosed by the arc BAC on the curve $y = x^3 + x + 2$ and the tangent from B to C. **3**

End of paper

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TASK #2

YEAR 12

Mathematics

Sample Solutions

Section	Marker
A	AF
B	RB
C	RD

1. a. i. $y = (5-2x)^9$

$y' = 9(5-2x)^8 . -2$

$= -18(5-2x)^8$

ii. $y = x + x^{\frac{1}{2}}$

$y' = 1 + \frac{1}{2}x^{-\frac{1}{2}}$

$= 1 + \frac{1}{2\sqrt{x}}$

b. $\int_2^5 f(x)dx = 12$

$\therefore \int_2^5 \frac{f(x)}{4} dx = \frac{1}{4}\int_2^5 f(x)dx$

$= \frac{1}{4} . 12$

$= 3$

c. $y = (x^2+3)^{-2}$

$y' = -2(x^2+3)^{-3} . 2x$

$= -4x(x^2+3)^{-3}$

$u = -4x$ $\quad v = (x^2+3)^{-3}$

$u' = -4$ $\quad v' = -3(x^2+3)^{-4} . 2x$

$= -6x(x^2+3)^{-4}$

$y'' = -4x . -6x(x^2+3)^{-4} + -4 . (x^2+3)^{-3}$

$= 24x^2(x^2+3)^{-4} - 4(x^2+3)^{-3}$

d. $\int_1^4 \frac{dt}{\sqrt{t}} = \int_1^4 t^{-\frac{1}{2}} dt$

$= \left[2t^{\frac{1}{2}}\right]_1^4$

$= 2(4)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}}$

$= 2$

e. $y = \frac{2}{x}$

$y = 2x^{-1}$

$y' = -2x^{-2}$

$y' = -\frac{2}{x^2}$

when $x = 1$

$y' = m_T = \frac{-2}{(1)^2}$

$m_T = -2$

$\underline{\underline{m_N = \frac{1}{2}}}$ $(m_1 \times m_2 = -1)$

when $x = 1$

$y = \frac{2}{1}$

$y = 2$

$\underline{\underline{\text{point } (1, 2)}}$

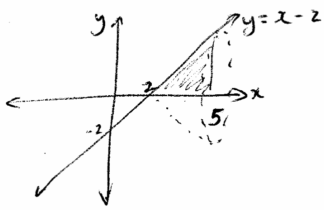
$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{2}(x - 1)$

$2y - 4 = x - 1$

$x - 2y + 3 = 0$

f.



$V = \pi \int_2^5 (x-2)^2 \, dx$

$= \pi \int_2^5 (x^2 - 4x + 4) \, dx$

$= \pi \left[\frac{x^3}{3} - 2x^2 + 4x \right]_2^5$

$= \pi \left[\frac{(5)^3}{3} - 2(5)^2 + 4(5) - \left(\frac{(2)^3}{3} - 2(2) + 4(2) \right) \right]$

$= 9\pi \text{ units}^3$

g. Probability that none are defective 0.8^{20}

Probability that at least one is defective $= 1 - 0.8^{20}$

$= 0.98847$

$\approx 98.85\%$

h. i. $\triangle XYC \mathbin{|||} \triangle ABC$

$$\frac{24}{32} = \frac{30}{30+a}$$

$$30+a = 30 \times \frac{4}{3}$$

$$30+a = 40$$

$$\underline{a = 10}$$

$\triangle ABX \mathbin{|||} \triangle XDC$

$$\frac{30}{a} = \frac{b}{12} = \frac{d}{c}$$

$$\frac{30}{10} = \frac{b}{12}$$

$$\underline{b = 36}$$

ii. $$\frac{d}{c} = \frac{30}{10}$$

$$\frac{c}{d} = \frac{1}{3}$$

$$\therefore \underline{c : d = 1 : 3}$$

i. i. $\frac{5}{20} = \frac{1}{4}$

ii. $\frac{5}{20} \times \frac{15}{19} + \frac{15}{20} \times \frac{5}{19} = \frac{15}{38}$ (probability of getting 1st prize only or 2nd prize only)

iii. $1 - \frac{15}{20} \times \frac{14}{19} = \frac{17}{38}$ $\left(1 - P(\text{no prize})\right)$

iv. $\frac{5}{20} \times \frac{4}{19} = \frac{1}{19}$

2.a.i. $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-3+(-1)}{2}, \frac{0+4}{2}\right)$

$= (-2, 2)$

ii. $m_{AB} = \frac{y_2-y_1}{x_2-x_1}$

$= \frac{4-0}{-1-(-3)}$

$= \frac{4}{2}$

$= 2$

since $\tan\theta = m$

$\therefore \tan\theta = 2$

iii. $y - y_1 = m(x - x_1)$

$y - 0 = 2(x - (-3))$

$y = 2x + 6$

iv. $m_{BC} = \tan(180 - \theta)$

$= -\tan\theta$

$= -2$

$y - y_1 = m(x - x_1)$

$y - 4 = -2(x - (-1))$

$y - 4 = -2x - 2$

$y = -2x + 2$

v. let $y = 0$

$0 = -2x + 2$

$2x = 2$

$x = 1$

$\therefore$ coordinates of C are $(1, 0)$

Hence, area of $\triangle ABC = \frac{1}{2} \times 4 \times 4$

$= 8$ units^2

vi. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d_{BC} = \sqrt{(-1-1)^2 + (4-0)^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$ units

Area of $\triangle ABC = 8 = \frac{1}{2} \times 2\sqrt{5} \times h$

$h = \frac{8}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$h = \frac{8\sqrt{5}}{5}$ units

which is the perpendicular distance from A to BC

b. since $AE = ED$ (E is the midpoint of AD)

$AF = FG$ (equal intercepts $BF // DG$)

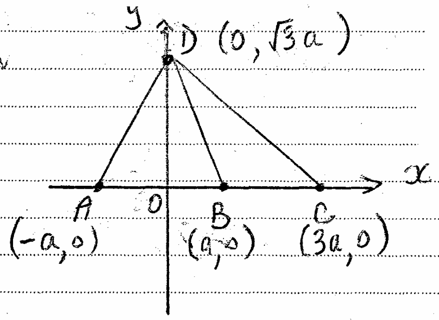
since $BD = DC$ (D is the midpoint of BC)

$FG = GC$ (equal intercepts $BF // DG$)

$\therefore AF = FG = GC$

(3)

Copy diagram into answer booklet.



(i) $\triangle DAB$ is equilateral

$d(AD) = \sqrt{(0--a)^2+(\sqrt{3}a-0)^2} = \sqrt{a^2+3a^2} = \sqrt{4a^2} = 2a$

$d(AB) = |-a| + a = 2a$

$d(BD) = \sqrt{(a-0)^2+(0-\sqrt{3}a)^2} = \sqrt{a^2+3a^2} = \sqrt{4a^2} = 2a$

Since $d(AB) = d(AD) = d(BD) = 2a$, $\triangle DAB$ is equilateral.

(2)

3 (ii) $\triangle BCD$ is isosceles with $BD = BC$.

$d(BC) = 3a - a = 2a$

$d(BD) = \sqrt{(a-0)^2 + (0-\sqrt{3}a)^2} = \sqrt{a^2 + 3a^2} = \sqrt{4a^2} = 2a$.

note $d(CD) = \sqrt{(3a-0)^2 + (0-\sqrt{3}a)^2} = \sqrt{9a^2 + 3a^2} = \sqrt{12}\,a$ ✓ 2

since $d(BC) = d(BD)$, $\triangle BCD$ is isosceles

(iii) $(BD)^2 = \frac{1}{3}(CD)^2$

$(2a)^2 = \frac{1}{3}\cdot(\sqrt{12}\,a)^2$

$4a^2 = \frac{1}{3} \times 12 \times a^2 = 4a^2$ ✓ 3

Q4 (i) at $x = 0$, $(0, y)$ ½ horizontal pt of inflexion ½ 2

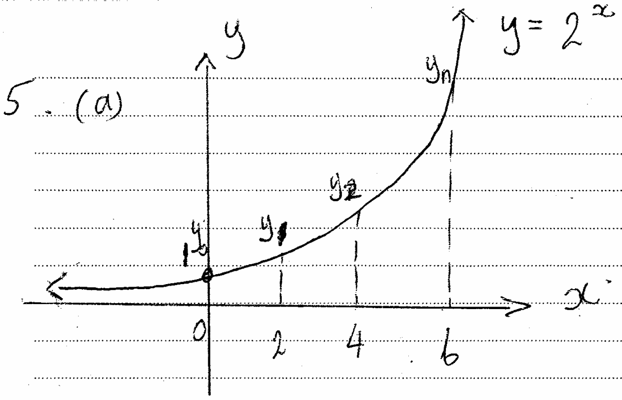
at $x = 5$, $(5, y)$ ½ MAX stat point ½

[$x = 3$ is an inflexion pt, where $f''(x) = 0$]

(ii) function decreasing when $x > 5$ 2

(iii) function concave down $x < 0$, $x > 3$ 2

Q5. (a)



①

$$\int_a^b f(x)\,dx \doteq \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right] \qquad h=\frac{b-a}{n}$$

So $$J=\int_0^6 2^x\,dx \doteq \frac{2}{2}\left[(2^0+2^6)+2(2^2+2^4)\right] \qquad h=\frac{6-0}{3}$$

$$=\left[1+64+2(4+16)\right]$$

$$= 65+40 = 105\,u^2$$ ③

(b) $$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^1 \cdot x^{\frac{1}{2}}\,dx = \int_0^4 x^{\frac{3}{2}}\,dx$$

$$\Rightarrow \left.\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^4$$

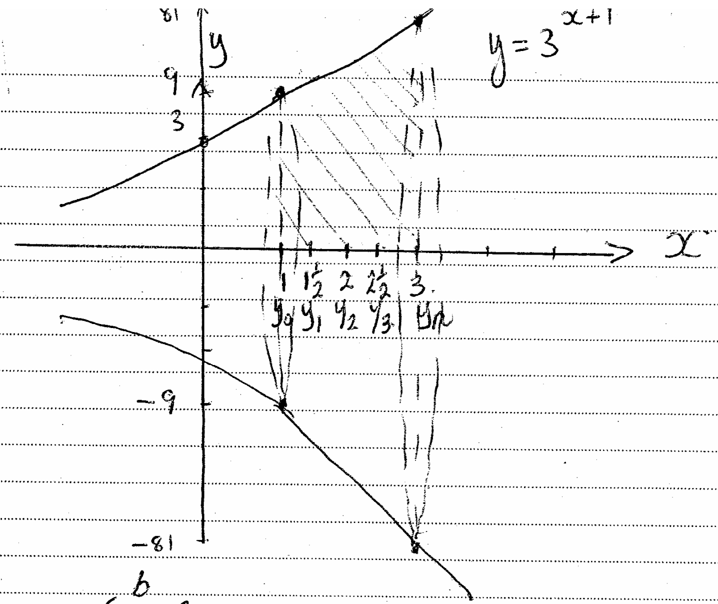
$$= \left.\frac{2x^{2\frac{1}{2}}}{5}\right]_0^4$$

$$= \frac{2}{5}\left[x^2 \cdot \sqrt{x}\right]_0^4 = \frac{2}{5}(16\times 2-0)$$

$$= \frac{64}{5} = 12.8 \; //$$ ②

Q6.

$y = 3^{x+1}$



$h = \frac{3-1}{4}$
$= \frac{2}{4} = \frac{1}{2}$.

$$V = \pi \int_a^b y^2 \, dx$$

and $A \doteqdot \int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \dots) + 2(y_2 + y_4 + \dots)\right]$

$h = \frac{b-a}{n}$

$$V = \pi \cdot \frac{\frac{1}{2}}{3}\left[(3^2)^2 + (3^4)^2 + 4\left((3^{2.5})^2 + (3^{3.5})^2\right) + 2\left((3^3)^2\right)\right]$$

$$= \frac{\pi}{6}\left[3^4 + 3^8 + 4 \times 3^5 + 4 \times 3^7 + 2 \times 3^6\right]$$

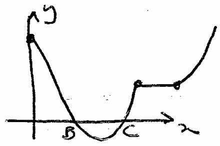
$$= \frac{\pi}{6}\left[81 + 6561 + 972 + 8748 + 1458\right]$$

$$= \frac{\pi}{6} \times 17820$$

$$= 9330.530 \ u^3$$

5

Q7



3

Not differentiable:

A $\quad \lim_{x\to 0^-} f'(x)$ does not exist

D $\quad \lim_{x\to x_D^-} f'(x) \neq \lim_{x\to x_D^+} f'(x)$

E $\quad$ (possibly) unless $\lim_{x\to x_E^+} f'(x) = 0$

Q8 $\quad y = 2x^3 - 9x^2 + 52$

(i) $y' = 6x^2 - 18x$

$y'' = 12x - 18$

For st pts: $y' = 0 \quad \therefore 6x(x-3) = 0$

$\therefore x = 0$ or $x = 3$ $\quad$ 2

When $x = 0$ $\; y'' = -18 \quad \therefore (0, 52)$ max TP

When $x = 3$ $\; y'' = 18 \quad \therefore (3, 25)$ min TP

(ii) For pt of inf: Consider $y'' = 0$

$\therefore 12x - 18 = 0$

$\therefore x = 1{\cdot}5$

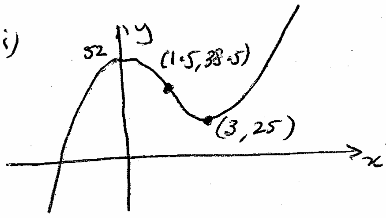
x	1	1·5	2
y''	-6	0	6
Concavity	down	–	up

2

$\therefore$ Change of concavity $\therefore$

$\therefore (1{\cdot}5, 38{\cdot}5)$ is pt of inflexion

(iii)



2

(iv) When $x = -1$, $y = 41$

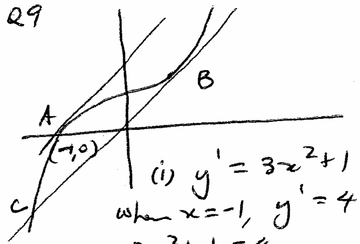
When $x = 2$, $y = 32$ $\quad$ 2

$\therefore$ For $-1 \le x \le 2$

min value $= 32$, max value $= 52$

(v) For $2x^3 - 9x^2 + 52 = k$ to have 3 solutions, $25 < k < 52$ $\quad$ 2

Q9



(i) $y' = 3x^2 + 1$

when $x = -1$, $y' = 4$

At B, $\quad 3x^2 + 1 = 4$

$\therefore 3x^2 = 3$

$\therefore x = \pm 1$

3

$\therefore$ B is $(1, 4)$

Eqn of tangent at B: $y - 4 = 4(x - 1)$

$y - 4 = 4x - 4$

$y = 4x$.

(ii) For C: $4x = x^3 + x + 2$

$x^3 - 3x + 2 = 0$

when $x = -2$: $x^3 - 3x + 2$

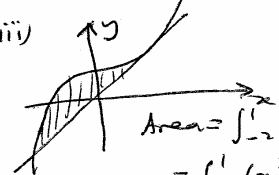
$= -8 + 6 + 2$

$= 0$ $\quad$ 2

$\therefore (-2, -8)$ is a pt of intersection

$\therefore$ C is $(-2, -8)$

(iii)



Area $= \int_{-2}^{1} (x^3 + x + 2 - 4x)\,dx$

$= \int_{-2}^{1} (x^3 - 3x + 2)\,dx$

$= \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x\right]_{-2}^{1}$

$= \left[\frac{1}{4} - \frac{3}{2} + 2\right] - \left[4 - 6 - 4\right]$

$= \left[+\frac{3}{4}\right] - [-6]$

$= 6\frac{3}{4}$

3